TOPOLOGY IN AND VIA LOGIC
HOMEWORK ASSIGNMENT 3

- Deadline: January 24 at 14:59.
- All exercises are worth the same points.
- The assignment can be completed in teams of up to two people.
- Good luck!

Compactness and Connectedness

Definition 1. Let X be a set, and $S \subseteq \mathcal{P}(X)$ closed under intersection, union, complement and contains $\emptyset$; we say that S is a *Boolean algebra*. We say that $F \subseteq S$ is an *S-filter* if:

(1) $X \in F$;
(2) If $U \in F$ and $U \subseteq V$ where $V \in S$, then $V \in F$;
(3) If $U, V \in F$ then $U \cap V \in F$.

Furthermore, we call it a *prime S-filter* if for each $U \in S$, either $U \in F$ or $X - U \in F$.

Note: Below you can use the fact that the Prime Filter Theorem, which we saw in class, holds for any Boolean algebra in the above condition.

Exercise 1. Let X be a topological space. Observe that

$$\mathsf{Clop}(X) = \{U \subseteq X : U \text{ is clopen}\}$$

is closed under intersection, union, complement and contains $\varnothing$. Thus, we can consider the collection of $\mathsf{Clop}(X)$-prime filters, denoted by $X^* = \mathit{Spec}(\mathsf{Clop}(X))$. We give this space a topology by specifying the following basis (you may assume without proof that this, indeed, is a basis for a topology on X^*):

$$\{\phi(U) : U \in \mathsf{Clop}(X)\} \text{ where } \phi(U) = \{F \in X^* : U \in F\}.$$

(1) Show that X^* is always a compact Hausdorff space. *Hint: For compactness, given* $X^* = \bigcup_{i \in I} \phi(U_i)$*, it might be helpful to consider*

$$\{U \in \mathsf{Clop}(X) \mid U \supseteq U_{i_0}^c \cap \cdots \cap U_{i_n}^c \text{ for some } \{i_0, \ldots, i_n\} \subseteq I\}.$$

(2) Show that the map $i : X \to X^*$ given by

$$i(x) := \{U \in \mathsf{Clop}(X) : x \in U\}$$

is well-defined.

BONUS EXERCISES (not compulsory)

Definition 2. Let X be a normal topological space. We say that X is *strongly zero-dimensional* if whenever A, B are disjoint closed sets, then there is some clopen set U such that $A \subseteq U$ and $B \subseteq X - U$.

Exercise 2. Let X be a topological space and X^* be defined as in Exercise 1.

(3) Assume that X is a strongly zero-dimensional space, and suppose that Z is some compact Hausdorff space, such that $f : X \to Z$ is a continuous function. Show that for $F \in X^*$ the map

$$\tilde{f}(F) := x_F, \text{ where } x_F \in \bigcap\{\overline{f[U]} : U \in F\}.$$

is a well-defined continuous map from X to Z, and has the property that $\tilde{f} \circ i = f$. (Hint: you can use the following fact without proof: for all distinct $u, v \in Z$, there exists open sets $U \in N(u)$ and $V \in N(v)$ such that $cl(U) \cap cl(V) = \varnothing$.)

(4) Conclude that for strongly zero-dimensional spaces we have that $X^* \cong \beta(X)$.