Topic Presentation Ideas

Introduction to Topology in and via Logic - 2025

Below you will find a number of topic presentation ideas. These are organized by section, and contain a brief outline of what the general theme is, and some ideas of how to structure your presentation. More precise ideas can be discussed later.

General Topological Questions

Whilst this course has covered many topics you will see in an introductory course in topology, even in the domain of general topology some natural concepts were left out. Hence it might be interesting to deepen these concepts.

- **Keeping you Close: Metric Spaces:** One of the key aspects of the real line is that you can keep track of the distance between two points on a space. Indeed, for products of the real line, the same holds. This is basically down to metric spaces. Introduce the notion of a metric space and show that all metric spaces are topological spaces. Discuss basic properties of metric spaces: topology, sequences in metric spaces, sequential continuity. Introduce the notion of a complete metric space, and define the completion of a metric space. In case you have time, you can look into the question of when a topological space is a metric space: give an example of a topological space that is not metrizable, and use Urysohn's Lemma to prove his metrization theorem or go even further by characterizing the class of metrizable topological spaces via the Nagata-Smirnov Metrization Theorem or the Smirnov Metrization Theorem.
- **Keeping you Close: Uniformities:** Sometimes the notion of a metric can be too strong for some applications. A weaker notion might be needed which nevertheless still captures the idea that two points are "close" in a strong, but qualitative, rather than quantitative sense. Define uniform spaces. Relate uniform spaces to metric and topological spaces. Then choose one: introduce uniform completions and Cauchy filters; or show that a topology is Tychonoff if and only if it can be given a uniform structure.
- **Keeping you Close: Order Topologies:** We have seen that when defining the standard (Euclidean) topology on $\mathbb{R}$, all we need to consider is its order: we generate the topology via the basis consisting of all open intervals. This idea generalizes most naturally to constructing

a topology on any given linear order, called its order topology, which allows for generalizing various theorems from analysis. Introduce the notion of an order topology and show how these topologies interact with other basic notions we have seen. For instance, you can show (some of) the following:

(i) for convex subsets, restricting the order topology is the same as taking the subspace topology;

(ii) all order topologies are Hausdorff (and even normal);

(iii) all convex subsets of dense linear orders with the l.u.b. property are connected;

(iv) all connected sets in the order topology are dense and have the l.u.b. property;

(v) if X, Y are order topologies and $f : X \to Y$ is order-preserving and surjective, then f is a homeomorphism

(vi) all order topologies are regular.

Then prove (1) a generalized topological intermediate value theorem and (2) a generalized topological extreme value theorem, and show how the usual Intermediate Value Theorem and Extreme Value Theorem on $\mathbb{R}$ obtain as special cases.

- **Projecting on Compact Hausdorff:** We have seen that the compact Hausdorff spaces are very nice structures. Investigate (two out of the three) further properties of these objects: given two compact Hausdorff spaces X, Y, there is a topology that can be put on X^Y, called the compact-open topology; a space is called *compactly generated* if it is coherent with the family of its compact subspaces - show they are all topological sums of compact Hausdorff spaces; show that if P is a compact Hausdorff and extremely disconnected space, $f : P \to X$ is a continuous map, and $e : E \to X$ is a surjective continuous map, then there exists a map $\overline{f} : P \to E$ such that $\overline{f} \circ e = f$.

Topology and Logic

We started this course by discussing an epistemic interpretation of topology, and later found that a very natural modal logic implements precisely the idea of being an interior operator, an equivalent way of defining topologies. How far does the logical description of topology go?

- **McKinsey and Tarski theorem for C-semantics:** McKinsey and Tarski proposed the interpretation of the unary modal operator $\Diamond$ as the closure operator *cl* of a topological space. This interpretations give the topological C-semantics for modal logic. C-semantics is closely related to the closure algebras. It is a landmark theorem of McKinsey and Tarski that S4 is the C-logic of any dense-in-itself metrizable space. The original McKinsey-Tarski result had an additional assumption that the space is separable. Rasiowa and Sikorski showed that this additional condition can actually be dropped, assumed AC. Give the definition of C-semantics, discuss the relation between topological spaces and closure algebra, and give a proof of McKinsey and Tarski theorem.

- **Abashidze-Blass theorem for d-semantics:** If we interpret the modality $\Diamond$ as the derived set operator d, then we obtain the d-semantics for modal logic. Esakia showed that $\mathsf{wK4}$ is the d-logic of all topological spaces. Moreover, Abashidze and Blass independently proved that $\mathsf{GL} = \mathsf{Log}_d(\alpha)$ for any ordinal α greater than ω^ω. Give the definition of d-semantics, discuss some of its properties (for example, compare the expressive power of C-semantics and d-semantics), and give a proof of Abashidze-Blass theorem.

- **Polyhedral modal logic:** It was shown that the set $Sub(P)$ of subpolyhedra of a polyhedron P forms a Boolean algebra closed under interior and closure. Polyhedral semantics is a way of interpreting modal formulas on polyhedra. This semantics was introduced by Bezhanishvili et.al recently. Give definitions of Polyhedral semantics and polyhedrally complete logics. Prove (some of) the followings: (i) Every poly-complete logic has the finite model property; (ii) The logics $\mathsf{S4.Grz.2}$ and $\mathsf{Grz.3}$ are poly-incomplete.

- **Topological products of modal logics:** In general, the product of two modal logics is the logic of the class of all products of structures of the corresponding logics. The study of products of Kripke frames and their modal logics was initiated by Segerberg and Shehtman. In 2006, van Benthem et.al introduced topological products of modal logics. The topological product of modal logics L_1 and L_2 containing $\mathsf{S4}$ is defined semantically as the modal logic of the class of all possible products of topological spaces of corresponding logics. It was shown that the topological product $\mathsf{S4} \times_t \mathsf{S4}$ is the same as the fusion $\mathsf{S4} \oplus \mathsf{S4}$. Define products and topological products of modal logics, discuss the relation between these two different products, and prove: (i) the complete logic of all topological products of topological spaces is weaker than $\mathsf{S4} \times \mathsf{S4}$; and (ii) $\mathsf{S4} \times_t \mathsf{S4} = \mathsf{S4} \oplus \mathsf{S4}$.

- **Modal logic of betweenness:** Let's now move on from topology to more 'rigid' spatial structures. For instance, consider *affine geometry*, where the major notion is a ternary notion of *betweenness* $\beta(xyz)$ between points. This says that point y lies in between x and z, allowing y to be one of these endpoints. Now define a binary betweenness modality $< B >$:

$$M, x \models < B > (\varphi, \psi) \text{ if and only if } \exists y, z(\beta(yxz) \wedge M, y \models \varphi \wedge M, z \models \psi).$$

This leads to very concrete spatial pictures, where modal logic play its role again. A modal logic of betweenness was studied by van Benthem and Bezhanishvili. Give definitions of affine geometry and modal logic of betweenness. Discuss affine bisimulation and expressive power of this logic. Give some examples.

- **Hyperspaces and Beyond:** A classic result by Leo Esakia constructed a form of "topological Kripke frames" by considering topologies on the power set of a topological space - so called "hyperspaces". Give an outline of the Vietoris topology on a general space, and its consequence for Stone spaces: the Vietoris constructions maps Stone spaces to Stone spaces. Show that there is a 1-1 correspondence between specific maps to the Vietoris space and descriptive general frames.

Set Theoretic Topology

What we have covered in this project is very much the core theory of general topology, but there was a set of topics which, for matters of space, were left out - the cardinality invariants, and other analysis of space which depend on the existence of specific combinatorial structures.

- **Weighing the Character of a Space:** On par with the separation axioms, we have axioms controlling the size of a space, and cardinal invariants corresponding to them. Introduce the notion of "weight", "density" and "character" of a topological space. Define first-countable, separable and second-countable spaces; give examples and counterexamples for these notions. Define the class of regular spaces. Then prove Urysohnn's embedding theorem.

- **Suslin's Real Problem:** Axiomatizing the reals has been a challenge throughout the history of mathematics, and many attempts were made to make precise what is special about them. The intimate connection of the reals with linear orders allows one to ask questions about this. Define a linearly ordered topological space. Provide a proof of the classical result showing that a linear order is complete, separable, dense and unbounded if and only if it is the real line. Introduce Suslin's problem. Introduce Martin's Axiom, and show that it resolves Suslin's problem in the positive.

- **Bor(r)eling in the Real Line:** When working with topological spaces, not all sets are open or closed. However, some sets can still be obtained by, in some sense "accessible operations". Describe the concept of a Borel set, and the corresponding Borel algebra. Introduce the notion of a Polish space as a natural setting to discuss Borel sets. Then prove that the Borel hierarchy does not collapse. If interested, discuss the connections with computability theory.

- The Mysteries of $\beta\omega$: The space $\beta\omega$, and the remainder $\beta\omega - \omega$ constitute a very complex structure, described as a "beast with three heads". Discuss p-points and their independence, to illustrate this. If interested, you can relate these filters to Ramsey-like theorems, where they appear naturally. Then provide (Engelking's topological) proofs of Parovicenko's theorems (without all details): in the presence of CH the space $\beta\omega - \omega$ is the unique space with the following properties:

  (i) Compact Hausdorff, which is crowded;

  (ii) Has weight equal to the cardinality of the continuum;

  (iii) Every two disjoint countable unions of closed sets have disjoint closures;

  (iv) Every non-empty countable intersection of open sets has non-empty interior.

  And in fact, this characterization is equivalent to CH.

Algebraic Topology

One aspect which is somewhat unsatisfactory about the notion of continuity is that, whilst it captures the idea that transformations are in some sense unbroken, this intuition is not truly captured by continuity. The most intuitive idea of continuity - of a continuous "deformation" of space, of a folding of space - can only be captured by other, more sophisticated concepts. These happen to almost always involve some algebra.

- **It is simple[x]:** Simplicial complexes constitute one of the simplest, yet most versatile classes of spaces available in mathematics, and they are used in everything, from modal logic, to category theory, to algebraic topology and set theory. Give the definition of a simplicial complex, and the basic properties of these spaces. Then define the basic idea of a chain group, boundaries and cycles, and show how this implements the idea of "higher-dimensional holes". Give many examples.

- **Deforming Space:** Homotopies are the right notion of continuous deformation of space. They can be given the structure of a group, which leads to the powerful methods of algebra. Introduce the notion of a homotopy, and give examples. Define the fundamental group. Define a covering map, and prove that Z is the fundamental group of the circle. If there is time, give a proof of the Borsuk-Ulam Theorem for $n = 2$.

Categorical Topology

Our whole emphasis in this course has been highly categorical. We have made use of an analogy - topology as a space of learning, or computability - which is strongly preffered by category theorists and computer scientists alike. However, we have strived to keep the presentation close to a classical presentation, mostly through a strong concern with point-set notions. What happens if you relinquish that?

- **Let's get Stoned:** Often we think of topological spaces as sets with extra structure; however, it is also possible, in certain cases, to think of them as special kinds of algebras. Introduce Stone duality through the lens of capturing space through algebra, and contrast with Stone's original motivation (capturing algebra through topology). Then discuss the basics of locale theory: completely prime filters, spatial locales and sober spaces.

- **Topping Topoi:** Topoi constitute a wide range of structures with wide applications. But even their definition can sometimes elicit fear in the heart of people. Define a sheaf over topological spaces, and illustrate, with examples of sheaves of continuous functions, that these are natural structures. Then define a (localic) Grothendieck topos as this kind of structure. Give a brief sketch of the sheafification procedure.

Philosophy and Topology

In our approach we emphasized the intuitions of epistemology to motivate our work. Many of these intuitions can be made more thoroughly connected through some formalization of this in the form of topological epistemic logic. But this does not exhaust the connections between philosophy and topology. Mereology has emerged in the 21st century as a source of metaphysical questions, and as a deeply sophisticated formal theory, often rich in mathematical subtlety alongside its philosophical interest. In this setting, theories of location have given rise to specific approaches that share with topology an interest in space, occuppying space, amongst other spatial questions.

- **Topology of surprise:** Epistemic logic has been formalized by Hintikka within the framework of possible-world semantics in relational models. Bezhanishvili et.al. presented a topological epistemic logic, with modalities for knowledge (modeled as the universal modality), knowability (represented by the topological interior operator), and unknowability of the actual world. Two versions of surprises were studied there. Discuss the epistemic interpretation of topology and the formalizations of surprise by N. Bezhanishvili et.al, provide some examples and show some possible applications.

- **Misleading Defeaters and Defeating Misleaders:** Some recent work has connected epistemic logic and classic discussions about knowledge, justified belief, defeasibility and misleadingness. Present the basic arguments of the paper by Baltag, Bezhanishvili, Ozgun and Smets, and present the proposals of some topological models. Then analyse the logic. You should pick one particular epistemic concept - evidence, justified belief, knowledge - to focus on.
- **Space is a Soup:** Region-based theories of spatial structure are formalizations of space that depend on the notion of a "region" rather than points. These are often "thick" structures, and quite distinct from point-set topological spaces. Discuss the formalization by Casati and Varzi, providing examples and applications to philosophy of your interest.