SHORT ANSWER - TUTORIAL 1

INTRODUCTION TO TOPOLOGY IN AND VIA LOGIC - 2025

Short answers of some of the exercises in Tutorial 1.

Exercise 4. Let (X, τ) be a topological space and $S \subseteq X$. Show that the following hold:

(1) There exists an open set $int(S)$ such that (i) $int(S) \subseteq S$; and (ii) for all open set U, $U \subseteq S$ implies $U \subseteq int(S)$.

(2) There exists an closed set $cl(S)$ such that (i) $S \subseteq cl(S)$; and (ii) for all closed set U, $S \subseteq U$ implies $cl(S) \subseteq U$.

Answer. For (1), let $int(S) = \bigcup\{U \in \tau : U \subseteq S\}$. Since open sets are closed under arbitrary union, $int(S)$ is open. Clearly (i) and (ii) hold by definition.

Exercise 5. Let (X, τ) be a topological space and $A, B \subseteq X$. Prove the following statements:

(1) $A \subseteq cl(A)$ and $int(A) \subseteq A$.

(2) $cl(cl(A)) = cl(A)$ and $int(int(A)) = int(A)$.

(3) $cl(A) = X \setminus (int(X \setminus A))$ and $int(A) = X \setminus (cl(X \setminus A))$.

(4) $cl(A) \cup cl(B) = cl(A \cup B)$ and $int(A) \cap int(B) = int(A \cap B)$.

(5) If $A \subseteq B$, then $cl(A) \subseteq cl(B)$ and $int(A) \subseteq int(B)$.

Is $cl(A) \cap cl(B) = cl(A \cap B)$ or $int(A) \cup int(B) = int(A \cup B)$ true in general? Prove your answer.

Answer. We first prove half of (3) here. Since $int(X \setminus A) \subseteq X \setminus A$, we see $A \subseteq X \setminus (int(X \setminus A))$. Note that $X \setminus (int(X \setminus A))$ is closed and $cl(A)$ is the smallest closed set containing A, we have $cl(A) \subseteq X \setminus (int(X \setminus A))$. Similarly, since $X \setminus cl(A) \subseteq X \setminus A$ and $X \setminus cl(A)$ is open, we see $X \setminus cl(A) \subseteq int(X \setminus A)$, which entails $cl(A) \supseteq X \setminus (int(X \setminus A))$.

In general, $cl(A) \cap cl(B) \supseteq cl(A \cap B)$ holds, but the converse does not. Here is a counterexample: $\varnothing = cl((0,1) \cap (1,2)) \neq cl((0,1)) \cap cl((1,2)) = \{1\}$.

Definition 1. *Given a topological space (X, τ) and a point $x \in X$, we say that $V \in \mathcal{P}(X)$ is a* neighbourhood of x *if there is an open set U such that $x \in U \subseteq V$.*

Moreover, observe that if a neighbourhood V of a point x is open, the definition simplifies: V is an open neighbourhood of a point x if and only if $x \in V$ and V is open.[1]

Let $N(x)$ denote the set of all open neighbourhoods of x, i.e., $N(x) = \{U \in \tau : x \in U\}$.

Exercise 6. Suppose (X, τ) is a topological space and $S \subseteq X$. Then for all $x \in X$, the following are equivalent:

- x is in the closure of S, i.e., $x \in cl(S)$.
- All open neighbourhoods U of x have non-empty intersection with S, i.e.,

[1] In the literature, you will sometimes find that a neighbourhood is already required to be open. We do not adopt that convention, but simply speak of 'open neighbourhoods' when needed.

$$\forall U \in N(x)(U \cap S \neq \varnothing).$$

Answer. See Proposition 2.4.6 in the note.

Topological Semantics of Modal Logic

McKinsey and Tarski [1] proposed interpretations of the unary modal operator $\Diamond$ as the closure operator cl and the derived set operator d of a topological space. These interpretations give the topological C-semantics and d-semantics for modal logic.

In this section, we take a closer look at the topological C-semantics of modal logic.

Definition 2. *A topological model is a triple $\mathfrak{M} = (X, \tau, \nu)$ where (X, τ) is a topological space and $\nu : Var \to \mathcal{P}(X)$ a function called a valuation for X.*

A valuation ν is extended to the set Fm of all modal formulas by the following rules:

$$\nu(\bot) = \varnothing,\ \nu(\varphi \to \psi) = (X \setminus \nu(\varphi)) \cup \nu(\psi) \text{ and } \nu(\Diamond\varphi) = cl(\nu(\varphi)).$$

A formula φ is true at x in $\mathfrak{M}$, notation $\mathfrak{M}, x \models \varphi$, if $x \in \nu(\varphi)$. Note that by definition of the operator $cl : \mathcal{P}(X) \to \mathcal{P}(X)$, the following statements hold:

(a) $\mathfrak{M}, x \models \Box\varphi$ if and only if there is $U \in N(x)$ such that $\mathfrak{M}, y \models \varphi$ for all $y \in U$;
(b) $\mathfrak{M}, x \models \Diamond\varphi$ if and only if for all $U \in N(x)$, there is $y \in U$ such that $\mathfrak{M}, y \models \varphi$.

Exercise 7. Show that (a) and (b) in Definition 2 hold.
Answer. We show only (b) here. For any φ, we write $[\varphi]$ for the set $\{x \in X : \mathfrak{M}, x \models \varphi\}$. Then

$$\mathfrak{M}, x \models \Diamond\varphi \text{ iff } x \in [\Diamond\varphi] \text{ iff } x \in cl([\varphi]) \text{ iff } \forall U \in N(x)(U \cap [\varphi] \neq \varnothing) \text{ iff RHS},$$

where the third 'iff' follows from Exercise 6.

Recall that modal logic $\mathsf{S4}$ is defined to be $\mathsf{K} \oplus \{\Diamond\Diamond p \to \Diamond p, p \to \Diamond p\}$.
Exercise 9. Prove that every theorem of $\mathsf{S4}$ is valid. That is, given any $\varphi \in \mathsf{S4}$, $\mathfrak{M}, x \models \varphi$ for all topological model $\mathfrak{M} = (X, \tau, \nu)$ and $x \in X$.
Answer. It suffices to show that every axiom schema of $\mathsf{S4}$ is valid. For example, $\Box\top \leftrightarrow \top$ is valid since $int(X) = X$ for every topological space (X, τ); and $\Diamond\Diamond p \to \Diamond p$ is valid since $cl(cl(A)) \subseteq cl(A)$ for every topological space (X, τ) and $A \subseteq X$.

Exercise 10. Let (X, τ) be a topological space and $A \subseteq X$. We say that A is regular open if $int(cl(A)) = A$. Prove the following statements:

(1) if A is open, then $A \subseteq int(cl(A))$.
(2) $int(cl(A))$ is regular open.

Answer. We can always prove them topologically. But here we provide another proof. For (1), suppose A is open. Then $A = int(B)$ for some $B \subseteq X$. Note that $\Box p \to \Box\Diamond\Box p \in \mathsf{S4}$, by Exercise 9, $int(B) \subseteq int(cl(int(B)))$. Thus $A \subseteq int(cl(A))$.

(2) follows from the fact that $\Box\Diamond\Box\Diamond p \leftrightarrow \Box\Diamond p \in \mathsf{S4}$.

More on topological semantics: d-semantics

There are more than one semantics of modal logic based on topological spaces. The one we have seen in class is also called C-semantics. We now introduce the d-semantics as follows:

Definition 3. *Let $\mathcal{X} = (X, \tau)$ be a topological space and $x \in X$. A subset $Y \subseteq X$ is an* open neighborhood *of x if $x \in Y \in \tau$. Let $N(x)$ be the set of all open neighborhoods of x. For every subset $A \subseteq X$, let $\mathsf{d}(A)$ be the* derived set *of A, i.e.,*

$$\mathsf{d}(A) = \{x \in X : \forall U \in N(x)(U \cap (A \setminus \{x\}) \neq \varnothing)\}.$$

A topological model *is a triple $\mathcal{M} = (X, \tau, \nu)$ where $\mathcal{X} = (X, \tau)$ is a topological space and $\nu : \mathsf{Prop} \to \mathcal{P}(X)$ is a function which is called a* valuation *in $\mathcal{X}$. A valuation ν is extended to all modal formulas $\mathcal{L}$ as follows:*

$$\nu(\neg\varphi) = X \setminus \nu(\varphi),\ \nu(\varphi \vee \psi) = \nu(\varphi) \cup \nu(\psi) \text{ and } \nu(\Diamond\varphi) = \mathsf{d}(\nu(\varphi)).$$

*For each formula φ, φ is d-*true *at w in $\mathcal{M}$ (notation: $\mathcal{M}, w \models_{\mathsf{d}} \varphi$) if $w \in \nu(\varphi)$. We say that φ is d-valid if $\mathcal{M}, w \models_{\mathsf{d}} \varphi$ for all topological model $\mathcal{M}$ and point w in $\mathcal{M}$.*

Exercise 11. Try to understand the d-semantics given above and show:

(1) $\Diamond\Diamond p \to \Diamond p$ is not d-valid.
(2) $\Diamond\Diamond p \to \Diamond p \vee p$ is d-valid.

Answer. For (1), consider the model $\mathfrak{M} = (\{0, 1\}, \{\{0, 1\}, \varnothing\}, V)$ where $V(p) = \{0\}$. Then we see $\mathfrak{M}, 0 \models \Diamond\Diamond p \wedge \neg\Diamond p$. For (2), note that for each topological space $\mathcal{X} = (X, \tau)$ and $Y \subseteq X$, $\mathsf{d}(\mathsf{d}(Y)) \subseteq Y \cup \mathsf{d}(Y)$, we are done.

References

[1] McKinsey J. C. C., Tarski A., The algebra of topology, Annals of Mathematics **45**, 141–191 (1944)