

# TUTORIAL 1

## INTRODUCTION TO TOPOLOGY IN AND VIA LOGIC - 2025

### BASIC SET THEORY

**Exercise 1.** *The following results are used often in topology: Let  $X, Y$  be sets,  $f : X \rightarrow Y$  a function,  $S \subseteq X$ ,  $\{S_i : i \in I\} \subseteq \mathcal{P}(X)$ ,  $T \subseteq Y$  and  $\{T_j : j \in J\} \subseteq \mathcal{P}(Y)$ . Then*

- (1)  $f[\bigcup_{i \in I} S_i] = \bigcup_{i \in I} f[S_i]$ .
- (2)  $f[\bigcap_{i \in I} S_i] \subseteq \bigcap_{i \in I} f[S_i]$ .
- (3)  $f^{-1}[\bigcup_{j \in J} T_j] = \bigcup_{j \in J} f^{-1}[T_j]$ .
- (4)  $f^{-1}[\bigcap_{j \in J} T_j] = \bigcap_{j \in J} f^{-1}[T_j]$ .
- (5)  $f[S] \cap T = f[S \cap f^{-1}[T]]$ .

Furthermore, if  $f[\bigcap_{i \in I} S_i] = \bigcap_{i \in I} f[S_i]$  if  $f$  is injective. Prove them.

### BASIC TOPOLOGY

**Exercise 2.** *Recall that the Euclidean topology  $\tau_{Euc}$  on  $\mathbb{R}$  is defined as follows:*

*for all  $U \subseteq \mathbb{R}$ ,  $U \in \tau_{Euc}$  if and only if  $\forall z \in U \exists x, y \in U (z \in (x, y) \subseteq U)$ .*

*Verify that  $(\mathbb{R}, \tau_{Euc})$  is a topological space.*

**Exercise 3.** *Recall that the Cantor set is defined to be the set  $2^\omega$  of all binary sequences of length  $\omega$ . Let  $2^{<\omega}$  denote the set of all finite binary sequences. For all  $s \in 2^{<\omega}$  and  $t \in 2^\omega \cup 2^{<\omega}$ , we write  $s \triangleleft t$  if  $t \upharpoonright \text{dom}(s) = s$ . Intuitively,  $s \triangleleft t$  means that  $s$  is an initial subsequence of  $t$ . For each  $s \in 2^{<\omega}$ , we define the set  $C(s)$  by*

$$C(s) = \{t \in 2^\omega : s \triangleleft t\}.$$

*Let  $B = \{C(s) : s \in 2^{<\omega}\}$ . Verify that there is a unique topology  $\tau_{Can}$  on the Cantor set for which  $B$  is a basis.*

*The topological space  $(2^\omega, \tau_{Can})$  is called the Cantor space.*

### CLOSURE, INTERIOR AND NEIGHBOURHOODS

**Definition 1.** *Let  $(X, \tau)$  be a topological space. We say that a set  $U \in \mathcal{P}(X)$  is closed if its complement is open, i.e., if  $(X \setminus U) \in \tau$ .*

**Exercise 4.** *Let  $(X, \tau)$  be a topological space and  $S \subseteq X$ . Show that the following hold:*

- (1) *There exists an open set  $\text{int}(S)$  such that (i)  $\text{int}(S) \subseteq S$ ; and (ii) for all open set  $U$ ,  $U \subseteq S$  implies  $U \subseteq \text{int}(S)$ .*
- (2) *There exists a closed set  $\text{cl}(S)$  such that (i)  $S \subseteq \text{cl}(S)$ ; and (ii) for all closed set  $U$ ,  $S \subseteq U$  implies  $\text{cl}(S) \subseteq U$ .*

**Definition 2.** The sets  $\text{int}(S)$  and  $\text{cl}(S)$  in Exercise 4 are called the interior and the closure of  $S$ , respectively. Moreover, we see that a set  $S$  is closed if  $S = \text{cl}(S)$ , and open if  $S = \text{int}(S)$ . The operators

$$\text{int} : \mathcal{P}(X) \rightarrow \mathcal{P}(X), S \mapsto \text{int}(S)$$

and

$$\text{cl} : \mathcal{P}(X) \rightarrow \mathcal{P}(X), S \mapsto \text{cl}(S)$$

are called the topological interior and topological closure, respectively.

**Exercise 5.** Let  $(X, \tau)$  be a topological space and  $A, B \subseteq X$ . Prove the following statements:

- (1)  $A \subseteq \text{cl}(A)$  and  $\text{int}(A) \subseteq A$ .
- (2)  $\text{cl}(\text{cl}(A)) = \text{cl}(A)$  and  $\text{int}(\text{int}(A)) = \text{int}(A)$ .
- (3)  $\text{cl}(A) = X \setminus (\text{int}(X \setminus A))$  and  $\text{int}(A) = X \setminus (\text{cl}(X \setminus A))$ .
- (4)  $\text{cl}(A) \cup \text{cl}(B) = \text{cl}(A \cup B)$  and  $\text{int}(A) \cap \text{int}(B) = \text{int}(A \cap B)$ .
- (5) If  $A \subseteq B$ , then  $\text{cl}(A) \subseteq \text{cl}(B)$  and  $\text{int}(A) \subseteq \text{int}(B)$ .

Is  $\text{cl}(A) \cap \text{cl}(B) = \text{cl}(A \cap B)$  or  $\text{int}(A) \cup \text{int}(B) = \text{int}(A \cup B)$  true in general? Prove your answer.

**Definition 3.** Given a topological space  $(X, \tau)$  and a point  $x \in X$ , we say that  $V \in \mathcal{P}(X)$  is a neighbourhood of  $x$  if there is an open set  $U$  such that  $x \in U \subseteq V$ .

Moreover, observe that if a neighbourhood  $V$  of a point  $x$  is open, the definition simplifies:  $V$  is an open neighbourhood of a point  $x$  if and only if  $x \in V$  and  $V$  is open.<sup>1</sup>

Let  $N(x)$  denote the set of all open neighbourhoods of  $x$ , i.e.,  $N(x) = \{U \in \tau : x \in U\}$ .

**Exercise 6.** Suppose  $(X, \tau)$  is a topological space and  $S \subseteq X$ . Then for all  $x \in X$ , the following are equivalent:

- $x$  is in the closure of  $S$ , i.e.,  $x \in \text{cl}(S)$ .
- All open neighbourhoods  $U$  of  $x$  have non-empty intersection with  $S$ , i.e.,

$$\forall U \in N(x) (U \cap S \neq \emptyset).$$

There is a proof of this proposition in the note, but try to prove it yourself first : )

## TOPOLOGICAL SEMANTICS OF MODAL LOGIC

McKinsey and Tarski [1] proposed interpretations of the unary modal operator  $\diamond$  as the closure operator  $\text{cl}$  and the derived set operator  $\mathbf{d}$  of a topological space. These interpretations give the topological C-semantics and  $\mathbf{d}$ -semantics for modal logic.

In this section, we take a closer look at the topological C-semantics of modal logic.

**Definition 4.** A topological model is a triple  $\mathfrak{M} = (X, \tau, \nu)$  where  $(X, \tau)$  is a topological space and  $\nu : \text{Var} \rightarrow \mathcal{P}(X)$  a function called a valuation for  $X$ .

A valuation  $\nu$  is extended to the set  $Fm$  of all modal formulas by the following rules:

$$\nu(\perp) = \emptyset, \nu(\varphi \rightarrow \psi) = (X \setminus \nu(\varphi)) \cup \nu(\psi) \text{ and } \nu(\diamond \varphi) = \text{cl}(\nu(\varphi)).$$

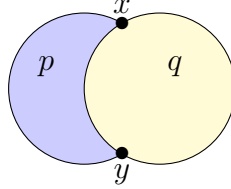
A formula  $\varphi$  is true at  $x$  in  $\mathfrak{M}$ , notation  $\mathfrak{M}, x \models \varphi$ , if  $x \in \nu(\varphi)$ . Note that by definition of the operator  $\text{cl} : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ , the following statements hold:

<sup>1</sup>In the literature, you will sometimes find that a neighbourhood is already required to be open. We do not adopt that convention, but simply speak of ‘open neighbourhoods’ when needed.

- (a)  $\mathfrak{M}, x \models \Box \varphi$  if and only if there is  $U \in N(x)$  such that  $\mathfrak{M}, y \models \varphi$  for all  $y \in U$ ;  
 (b)  $\mathfrak{M}, x \models \Diamond \varphi$  if and only if for all  $U \in N(x)$ , there is  $y \in U$  such that  $\mathfrak{M}, y \models \varphi$ .

**Exercise 7.** Show that (a) and (b) in Definition 4 hold.

**Exercise 8.** Consider the following topological model, where  $p$  is true in the blue area and its border line, and  $q$  is true in the yellow area and its border line.



- (1) Draw the region defined by the modal formula  $\Diamond p \vee \Box q$ .  
 (2) Find modal formulas that define the set  $\{x, y\}$ .

Recall that modal logic **S4** is defined to be  $\mathbf{K} \oplus \{\Diamond \Diamond p \rightarrow \Diamond p, p \rightarrow \Box p\}$ .

**Exercise 9.** Prove that every theorem of **S4** is valid. That is, given any  $\varphi \in \mathbf{S4}$ ,  $\mathfrak{M}, x \models \varphi$  for all topological model  $\mathfrak{M} = (X, \tau, \nu)$  and  $x \in X$ .

In fact, the converse of Exercise 9 also holds. Thus we have

**Theorem 5.** **S4** is the modal logic of all topological spaces.

**Exercise 10.** Let  $(X, \tau)$  be a topological space and  $A \subseteq X$ . We say that  $A$  is regular open if  $\text{int}(\text{cl}(A)) = A$ . Prove the following statements:

- (1) if  $A$  is open, then  $A \subseteq \text{int}(\text{cl}(A))$ .  
 (2)  $\text{int}(\text{cl}(A))$  is regular open.

#### MORE ON TOPOLOGICAL SEMANTICS: D-SEMANTICS

There are more than one semantics of modal logic based on topological spaces. The one we have seen in class is also called C-semantics. We now introduce the d-semantics as follows:

**Definition 6.** Let  $\mathcal{X} = (X, \tau)$  be a topological space and  $x \in X$ . A subset  $Y \subseteq X$  is an open neighborhood of  $x$  if  $x \in Y \in \tau$ . Let  $N(x)$  be the set of all open neighborhoods of  $x$ . For every subset  $A \subseteq X$ , let  $\mathbf{d}(A)$  be the derived set of  $A$ , i.e.,

$$\mathbf{d}(A) = \{x \in X : \forall U \in N(x)(U \cap (A \setminus \{x\}) \neq \emptyset)\}.$$

A topological model is a triple  $\mathcal{M} = (X, \tau, \nu)$  where  $\mathcal{X} = (X, \tau)$  is a topological space and  $\nu : \text{Prop} \rightarrow \mathcal{P}(X)$  is a function which is called a valuation in  $\mathcal{X}$ . A valuation  $\nu$  is extended to all modal formulas  $\mathcal{L}$  as follows:

$$\nu(\neg \varphi) = X \setminus \nu(\varphi), \nu(\varphi \vee \psi) = \nu(\varphi) \cup \nu(\psi) \text{ and } \nu(\Diamond \varphi) = \mathbf{d}(\nu(\varphi)).$$

For each formula  $\varphi$ ,  $\varphi$  is d-true at  $w$  in  $\mathcal{M}$  (notation:  $\mathcal{M}, w \models_{\mathbf{d}} \varphi$ ) if  $w \in \nu(\varphi)$ . We say that  $\varphi$  is d-valid if  $\mathcal{M}, w \models_{\mathbf{d}} \varphi$  for all topological model  $\mathcal{M}$  and point  $w$  in  $\mathcal{M}$ .

**Exercise 11.** Try to understand the d-semantics given above and show:

- (1)  $\Diamond \Diamond p \rightarrow \Diamond p$  is not d-valid.  
 (2)  $\Diamond \Diamond p \rightarrow \Diamond p \vee p$  is d-valid.

## REFERENCES

- [1] McKinsey J. C. C., Tarski A., The algebra of topology, *Annals of Mathematics* **45**, 141–191 (1944)