It is simple[x]

Simplicial homology

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Overview

Why we need homology?

Drawbacks of the fundamental group:

1. Computationally difficult;
2. Some "holes" fail to be detected, for example we have:

$$\pi_1(\mathbb{R}^3) = 0 \text{ and } \pi_1(\mathbb{R}^3 \setminus \mathring{\mathbb{D}}^3) = 0$$

Why we need homology?

One possible solution is to employ *Higher homotopy groups*, those, however, come with their own problems:

1. Sometimes detects "holes" of dimensions higher than desired. For example:

$$\pi_3(S^2) = \mathbb{Z}$$

2. Even harder to compute.

Homology provides a computationally (relatively) feasible procedure for detecting holes that stops counting at the desired dimension.

Manifolds

A n-dimensional manifold is a topological space with the property that each point has a neighborhood that is homeomorphic to an open subset of n-dimensional Euclidean space.

A topological n-dimensional manifold with boundary has a neighborhood homeomorphic to either $\mathbb{R}^n$ or the so called n-dimensional half-space $\mathbb{H}^n$.

Manifolds

Notice that n-dimensional manifolds are *locally euclidean*.

1. 1-manifold: the real line $\mathbb{R}$, the interval $(-1, 1)$ and the circle S^1
2. 2-manifold: an arbitrary surface Σ

Every topological manifold is homeomorphic to a closed subset of $\mathbb{R}^n$ for $n \in \mathbb{N}$ sufficiently large.

Triangulation

In homology, one aims to characterize properties of complex spaces without having to mention the complex spaces themselves. *Triangulation* gives us a way to do so:

A *Triangulation* of a topological n-manifold M is a homeomorphism of M to the polyhedron of a finite n-dimensional simplicial complex.

We can conceive each manifold as a structured collection of generalized triangles, named *simplices* which carry all the relevant data of the original space.

Standard simplex

The *Standard n-simplex* is defined as the set:

$$\Delta^n := \{(t_0, \ldots, t_n) \in I^{n+1} \mid t_0 + \cdots + t_n = 1\}$$

for each integer $n \geq 0$.

In other words, Δ^n is the convex hull of its $(n+1)$-vertices, which are the standard basis vectors of $\mathbb{R}^{n+1}$.

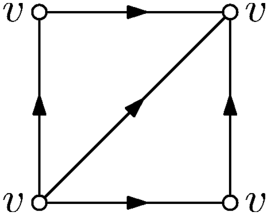
Standard simplices, examples

We readily see that:

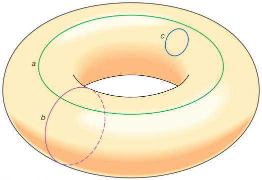
- the standard $0-$ *simplex* is just the point: $\{1\} \subseteq \mathbb{R}$;
- the standard $1-$ *simplex* is a *path* i.e., a line in $\mathbb{R}^2$ homeomorphic to $[0,1]$;
- the standard $2-$ *simplex* is the region in $\mathbb{R}^3$ bounded by a triangle.
- etc.

by *triangulating* any surface Σ, we conceive it as the union of copies of Δ^2 such that the intersection of any Δ^2_i, Δ^2_j is either $\emptyset$ or a copy of either Δ^1 or Δ^0.

Triangulation: example



(a) A triangled torus



(b) A torus!

Simplicial complexes

A simplicial complex K consists of two sets V and S called the set of vertices and the set of simplices respectively.

The elements of S are nonempty finite subsets of V and $\sigma \in S$ is called an n-simplex of K if it has $n+1$ elements. Further, the following is required:

1. Every vertex $v \in V$ gives rise to a 0-simplex in K, i.e $\{v\} \in S$
2. If $\sigma \in S$ then every subset $\sigma' \subset \sigma$ is also an element of S

Faces and boundary faces

Faces

For any n-simplex σ, we say σ' is a face of σ if $\sigma' \subset \sigma$.
If, in particular, σ' is a $n-1$ face of σ (i.e., if it has n vertices), we say σ' is a boundary face of σ.

With this in mind, condition 2) above simply means that a simplicial complex always contains all of its boundary faces.

The K-th boundary face

We denote the k_{th} boundary face of a simplex $\sigma = \{v_0, ..., v_n\}$ as $\partial_{(k)}\sigma$. Intuitively, this is the convex hull of $\{v_0, ..., \hat{v}_k, ..., v_n\}$ where $\hat{v}_k$ denotes the exclusion of the k-th vertex.

The k_{th} boundary face of a simplex Δ^n is: $\partial_{(k)}\Delta^n = \{t_k = 0\} \subset \Delta^n$.
Notice that $\partial_{(k)}\Delta^n \cong \Delta^{n-1}$ through the map:

$$(t_0, ..., t_{k-1}, 0, t_{k+1} ..., t_n) \mapsto (t_0, ..., t_{k-1}, t_{k+1} ..., t_n)$$

Polyhedra

To embed spaces in simplicial complexes, we need to equip the latter with a topology. For $K = (V, S)$, and any $\sigma \in S$, we define a subspace of $\mathbb{R}^N$ for each $\sigma = \{v_{i1}, \ldots, v_{ik}\}$ as follows:

$$\Delta_\sigma = \{(t_1, \ldots, t_N) \in I^N | \, v_{i1} + \ldots + v_{ik} = 1\,,\ v_j = 0 \text{ for all } v_j \notin \sigma\}$$

Δ_σ is here homeomorphic to the original simplex σ_k but it has now an obvious topology as a subspace of $\mathbb{R}^N$.

Polyhedra

For K a simplicial complex, the polyhedron of K is then the space

$$\mid K \mid := \bigcup_{\sigma \in S} \Delta_\sigma \subset \mathbb{R}^n$$

A triangulation of a topological n-manifold M is a homeomorphism of M to the polyhedron of a finite n-dimensional simplicial complex.

Polyhedra: example

Let $V = \{1, 2, 3, 4\}$ and S is the down-set of $\{A, B\}$ with $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$.

$$|K| \cong \mathbb{R} \times \mathbb{R}$$

Orientation

In the following, we are going to make use of “boundary faces” to compute the homology group of surfaces. However, to ensure that a surface really has boundaries we are going to request that it is *orientable*.

For our purposes, an orientation of a surface is a choice of which loop around any point should be labeled “clockwise” or “counterclockwise”.

Orientation

For a simplicial complex $K = (V, S)$ an orientation of an n-simplex $\sigma \in S$ for $n \geq 1$ is a an equivalence class of orderings of the vertices $v \in \sigma$, where two orderings are defined to be equivalent if and only if they are related to each other by even permutation.
An orientation of a 0-simplex is defined simply as assignment of the number $+1$ or -1 to the vertex.

This means that every point has either a positive or a negative orientation.

Orientation

Given an oriented n-simplex for $n \geq 2$ with vertices $v_0, \ldots, v_n$ ordered accordingly, the induced boundary orientation of its kthface $\partial_{(k)}\sigma$ is defined as the same ordering of its vertices if k is even and otherwise it is defined by any odd permutation of this ordering. For $n = 1$, the boundary orientation are defined by assigning $+1$ to $\partial_{(0)}\sigma = \{v_1\}$ and -1 to $\partial_{(1)}\sigma = \{v_0\}$.

Oriented triangulation

An oriented triangulation of a closed surface Σ is a triangulation $\Sigma \cong | K |$ together with a choice of orientation for each 2-simplex in the complex K such that for every 1-simplex $\Delta_\sigma \in K$, the two induced boundary that inherits as a boundary face of two distinct 2-simplices are opposite.

Oriented triangulation: example

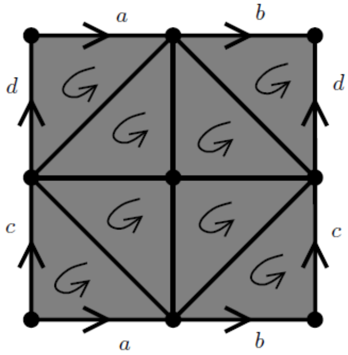


Figure: An oriented triangulation of the 2-Torus

Un-orientable objects

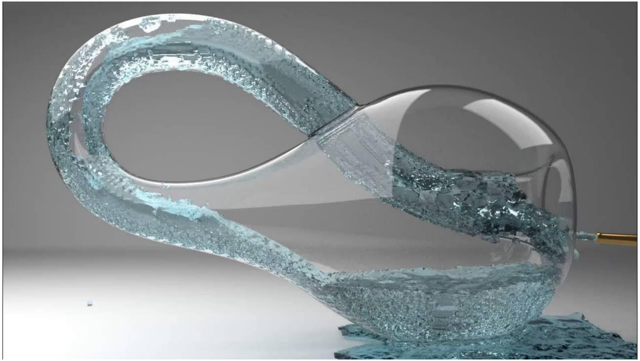


Figure: The Klein bottle

The main idea

Take a simplicial complex $K = (V, S)$ with associated polyhedron $X :=| K |$ and for each integer $n \geq 0$ and let $S_{(n)} \subset S$ denote the set of n-simplices. As auxiliary data we also a fix a group G. We choose G to be either $\mathbb{Z}$ or $\mathbb{Z}_2$.

n-chains

The group of n-chains in K with coefficients in G is the abelian group

$$C_n(K;G) := \bigoplus_{\sigma \in S_n} G,$$

whose elements can be written as finite sums $\Sigma_i a_i \sigma_i$ with $a_i \in G$ and $\sigma_i \in S_{(n)}$ with the group operation defined by:

$$\Sigma_i a_i \sigma_i + \Sigma_i b_i \sigma_i = \Sigma_i (a_i + b_i)\sigma_i$$

n-chains (geometrically)

In general, we only require G to be abelian, however, by setting $G = \mathbb{Z}$ we can consider $C_n(K, \mathbb{Z})$ as the union of all n-simplices in $|K|$ with orientation determined by the coefficients in $\mathbb{Z}$ i.e., ± 1.

The boundary operator

We can conceive the n_{th}-simplicial homology group of a space X as the free abelian group of the n-dimensional simplices in X *without boundaries*, quotiented by the boundaries of $n+1$-dimensional simplices in X.

Recall the notation $\partial_{(k)}\sigma$ to denote the k_{th}-boundary face of a simplex σ. We can express the *boundary of a simplex* σ as:

$$\partial\sigma = \sum_{k=0}^{n} \epsilon_k \partial_{(k)}\sigma \in C_{n-1}(K, \mathbb{Z})$$

Where ϵ is 1 if $\partial_{(k)}\sigma$'s chosen orientation is the same as the orientation it inherits from σ and -1 otherwise.

The boundary operator

The ∂_n operator determines induces a unique group homomorphism:

$$\partial_n : C_n(K, G) \to C_{n-1}(K, G) \text{ via the mapping } \sum_i a_i \sigma_i \mapsto \sum_i a_i(\partial \sigma_i)$$

Moreover, notice that the boundary of a $n-$simplex is always a $n-1$-simplex with empty boundary. So we set:

$$\partial_{n-1} \circ \partial_n = 0 \text{ for all } n \in \mathbb{N}$$

Cycles and boundaries

Cycles

We call the subgroup $\ker \partial_n \subset C_n(K; G)$ the group of *n*-**cycles** or, equivalently, the closed *n*-chains.

Boundaries

We call the elements of the subgroup $\operatorname{im} \partial_{n+1} \subset C_n(K; G)$ **boundaries**.

The simplicial homology

The fact that $\partial_{n-1} \circ \partial_n = 0$ implies that $im(\partial_{n+1})$ is a subgroup of ∂_n since, in particular, it a subset of the kernel.

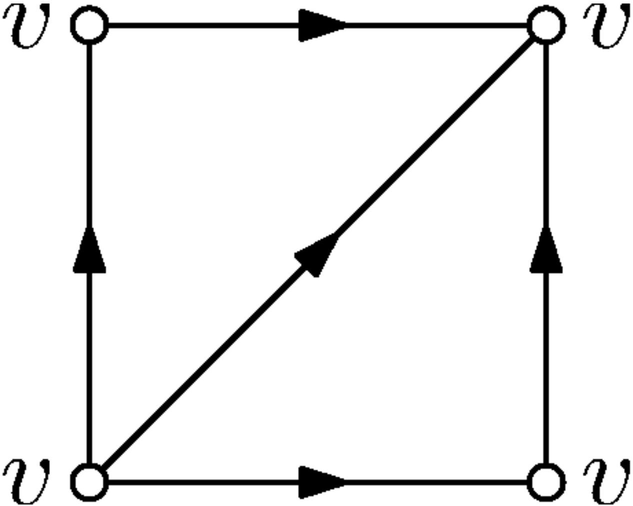
Since all $C_n(K, G)$ are abelian, all subgroups are normal, so we can consider the quotiented subgroups to obtain an abelian group.

Simplicial homology

The nth **simplicial homology** group of the complex K with coefficients in G is

$$H_n^{\Delta}(K; G) := \ker \partial_n / \operatorname{im} \partial_{n+1}$$

Torus!



The *Hauptvermutung* and Singular Homology

Theorem

For any simplicial complex K the simplicial homology groups $H_n^{\Delta}(K;G)$ depend (up to isomorphism) on the topological space $X = |K|$, i.e. the polyhedron of K, but not on the complex K itself.

The request that simplicial homology groups be independent from the choice of the polyhedral complex was founded on a conjecture, the **Hauptvermutung**, stating that every pair of triangulations of a given space could be turned into one by subdivision. However, the Hauptvermutung was proven false in the '60s and the proof of the above theorem had to pass through the development of *Singular Homology*.

The idea of singular homology

The main idea of singular homology is to find a topological invariant of a space X that correspond to its simplicial homology whenever X is a polyhedron, but does not explicitly make use of simplicial complexes.

To do so we need to consider not simplices, but continuous maps from simplices to our space of choice.

The idea of singular homology

Singular n-simplices

Given a topological space X, a *singular* n-simplex is a continuous map $\sigma; \Delta^n \to X$. We denote the set of singular n-simplices for X as $\mathcal{K}_n(X)$.

Singular n-chain groups

The singular n-chain group of X is defined as:

$$C_n(X; G) = \bigoplus_{\sigma \in \mathcal{K}_n(X)} G$$

The idea of singular homology

Singular boundary map

The singular boundary map $\partial\sigma : C_n(X;G) \to C_{n-1}(X;G)$ is defined as the restriction of σ to each of the faces:

$$\sum_{k=0}^{n}(-1)^k(\sigma \restriction_{\partial:(k)\Delta^n})$$

We can then define the singular n-th homology group of a chain complex.

Singular n-th Homology group

$$H_n(X;G) = \ker\partial_n \setminus \mathrm{im}\partial_{n+1}.$$

The idea of singular Homology

Thank you!