Topology project, 2nd lecture

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Plan for the day

- A few announcements
- Recap
- New stuff
- Break from 11h45-12h
- More new stuff

Announcements:

- First assignment has been published
- Updated schedule for next week, see website
- Presentations: it is advised to start finding a group and thinking about a potential topic, and then let us know your thoughts and findings during next week. If you don't have any topic ideas, no worries, just let us know

Recap: topological spaces

Definition (topological space)

Let X be a set. $\tau \subseteq \mathcal{P}(X)$ is a *topology on* X :iff

(O1) $\varnothing$ and X are in τ; i.e., $\varnothing \in \tau$ and $X \in \tau$.

(O2) τ is closed under *arbitrary* unions.

(O3) τ is closed under *finite* intersections.

Terminology:

- τ is a *topology* on X
- (X, τ) is a *topological space* (or simply: X is a top. sp.)
- $U \in \tau$ is *open*

Question: Given a set X, is there a unique topology τ on X?

Logic	Topology
Epistemic worlds/situations/models/etc.	Points, $x \in X$
Verifiable propositions	Open sets, $U \in \tau$

Recap: bases and subbases

Definition (basis and subbasis)

Given a top. sp. (X, τ), $\mathcal{B} \subseteq \tau$ is a *basis for the topology* τ :iff $\forall U \in \tau$, $\exists (V_i)_{i \in I} \subseteq \mathcal{B}$ s.t.

$$U = \bigcup_{i \in I} V_i.$$

Further, $\mathcal{S} \subseteq \tau$ is a *subbasis for the topology* :iff $\{\bigcap_{V \in M} V \mid M \subseteq \mathcal{S}, M \textit{is finite}\}$ forms a basis for the topology.

Terminology: Given a (sub)basis $\mathcal{B} \subseteq \tau$, we call members $U \in \mathcal{B}$ *(sub)basic opens*.

Proposition

Let X be a set and $C \subseteq \mathcal{P}(X)$ a collection of sets. Then there is a (unique) topology on X for which C is a subbasis.

Moreover, if (1) C covers X (i.e., $\bigcup_{U \in C} U = X$) and (2) C is closed under binary intersections, then there is a (unique) topology on X for which C is a basis.

Question: Given a ts (X, τ), is there a unique (sub)basis $\mathcal{B}$ for τ?

Recap: comparing topologies

Definition

Let X be a set, and τ and τ' two topologies on this set. We say that τ is a *coarser topology* than τ' if $\tau \subseteq \tau'$. Conversely, we say that τ' is *finer* than τ.

(Highly useful) lemma

Suppose X is a set with two topologies τ and τ', and $\mathcal{B}_\tau$ and $\mathcal{B}_{\tau'}$ are bases for these topologies, respectively. Then $\tau \subseteq \tau'$ iff for all points $x \in X$ and all basic τ-open $U \in \mathcal{B}_\tau$ containing x, there is some basic τ'-open $U' \in \mathcal{B}_{\tau'}$ such that $x \in U' \subseteq U$.

Finally, we were just about to exemplify the use of this lemma:

Comparing tops on $\mathbb{R}$: $\tau_F \subsetneq \tau_E$

Euclidean top, τ_E, and top, τ_F, gen. by basis $\{(l, \infty) \mid l \in \mathbb{R}\}$.

Questions?

Generating New Topologies: Subspaces

Definition (subspace)

Let (X, τ) be a ts and $S \subseteq X$. We denote by τ_S the *subspace topology on S* defined as

$$\tau_S := \{U \cap S \mid U \in \tau\}.$$

Terminology: (S, τ_S) is a *subspace* of (X, τ).

Lemma (subspace basis)

Let (X, τ) be a ts with a basis $\mathcal{B}$, and let $S \subseteq X$. Then the set

$$\mathcal{B}_S = \{U \cap S : U \in \mathcal{B}\}$$

is a basis for τ_S.

Proof

See blackboard.

Subspace top on $\mathbb{Z} \subseteq \mathbb{R}$

See blackboard.

Generating New Topologies: Finite Products

Definition (product top)

Let X and Y be ts. We define a topology on the product $X \times Y$, called the *product topology*, as follows: a set $U_X \times U_Y \subseteq X \times Y$ is basic open :iff U_X is open in X and U_Y is open in Y.

Proposition (subspaces and products commute)

Suppose X and Y are ts; $S_X \subseteq X$; and $S_Y \subseteq Y$. Then first constructing the product topology $X \times Y$ and then constructing the subspace topology $S_X \times S_Y \subseteq X \times Y$ is the same as first constructing the subspace topologies $S_X \subseteq X$ and $S_Y \subseteq Y$ and then taking their product $S_X \times S_Y$.

Proof

See blackboard.

Lemma

Let X and Y be ts with bases $\mathcal{B}_X$ and $\mathcal{B}_Y$. Then $\{U_X \times U_Y \mid U_X \in \mathcal{B}_X, U_Y \in \mathcal{B}_Y\}$ forms a basis for the product topology on $X \times Y$.

Proof of lemma is an exercise.

Closed sets

Definition

Let (X, τ) be a ts. We say that a set $U \in \mathcal{P}(X)$ is *closed* if its complement is open; i.e., if $(X - U) \in \tau$.

Proposition

Let (X, τ) be a ts. Then:

(C1) X and $\varnothing$ are closed sets.

(C2) Arbitrary intersections of closed sets are closed.

(C3) Finite unions of closed sets are closed.

Proof.

Follows from the complement operator taking unions to intersections (and vice versa). $\square$

Lemma

Suppose (S, τ_S) is a subspace of (X, τ). Then a set $U \in \mathcal{P}(S)$ is closed in S iff there is some closed set V in X (i.e., $(X - V) \in \tau$) such that $U = V \cap S$.

Proof of lemma is an exercise.

Epi. int.: what are the closed sets?

Falsifiable propositions

Recall:

($\exists\neg WS$) *There is a non-white swan.*

($\forall WS$) *All swans are white.*

Definition

A proposition P is *verifiable* :iff whenever P is true at a world x (i.e., $x \in [\![P]\!]$), it is possible to verify P at x (i.e., verify $x \in [\![P]\!]$).

Dually, we can define:

Definition

A proposition P is *falsifiable* :iff whenever P is false at a world x (i.e., $x \notin [\![P]\!]$), it is possible to falsify P at x (i.e., falsify $x \in [\![P]\!]$).

Then a proposition is falsifiable iff its negation is verifiable. I.e., the closed sets are precisely the falsifiable propositions!

Open, closed, both and neither

NB: *Unlike a door, we do <u>not</u> have: $S \subseteq X$ is closed iff S is not open*
Sets can also be both open and closed (shortened clopen); or neither open nor closed.

Using our epistemic intuition, we can make very good sense of this:

	Verifiable (open)	Falsifiable (closed)
All swans are white		x
Some swan is non-white	x	
It is raining outside	x	x
JFK's last thought was "What is the OTL?"		

Closure and interior

Definition

Let X be a ts and $S \subseteq X$. We denote by

$$cl(S) := \overline{S} := \bigcap \{S \subseteq C \mid C \textit{ is closed}\}$$

the *closure of* S, which is the smallest closed set K such that $S \subseteq K$. We denote by

$$int(S) := \bigcup \{U \subseteq S \mid U \textit{ is open}\}$$

the *interior of* S, which is the largest open set K such that $K \subseteq S$.

Observation: Using this def., we get:

(C) $S \subseteq X$ is closed iff $S = \overline{S}$;

(O) $S \subseteq X$ is open iff $S = int(S)$.

Neighbourhoods

Definition

Given a ts (X, τ) and $x \in X$, $V \subseteq X$ is a *neighbourhood* of x :iff there is an open set U such that $x \in U \subseteq V$.
Observe that if a neighbourhood V of a point x is open, the definition simplifies: V is an open neighbourhood of a point x iff $x \in V$ and V is open.

Proposition

Suppose X is a ts and $S \subseteq X$. Then TFAE for a point $x \in X$:

- x is in the closure of S; i.e., $x \in cl(S)$.
- All open neighbourhoods U of x have non-empty intersection with S; i.e., $U \cap S \neq \varnothing$.

Proof.

By contrapositions, see blackboard. $\square$

Epistemic intuition: summary

Logic	Topology
Epistemic worlds/situations/etc.	Points, $x \in X$
Verifiable propositions	Open sets, $U \in \tau$
Falsifiable propositions	Closed sets, $U^C \in \tau$
Verifiable propositions true at x	Open neighbourhoods U of x
(Sub)basic verifiable propositions	(Sub)basic opens

A teaser for Monday …

Continuous functions

Definition

Let $f : X \to Y$ be a function between topological spaces. We say that f is *continuous* :iff for all $U \subseteq Y$ open in Y, the preimage

$$f^{-1}[U] := \{x \in X \mid f(x) \in U\}$$

is open in X.

What a seemingly weird definition of continuity … Let's use this proposition to see how it actually does agree with our intuition of continuity:

Proposition

Let $f : X \to Y$ be a function between topological spaces. Then TFAE:

(i) f is continuous

(ii) For every $S \subseteq X$: $f\left(\overline{S}\right) \subseteq \overline{f(S)}$, i.e., if $x \in cl_X(S)$ then $f(x) \in cl_Y(f(S))$

Interpretation: For $S \subseteq X$ and $x \in X$, we say that x *is close to* S :iff $x \in cl(S)$.
Then f is continuous iff

for every $S \subseteq X$, f maps points close to S to points close to $f(S)$.

Open maps (and why they do not formalise continuity)

Definition (Open map)

Let (X, τ_X) and (Y, τ_Y) be ts, and $f : X \to Y$ a map between them. We say that f is *open* if for every open U in X, its image $f[U] = \{f(x) \in Y \mid x \in U\}$ is open in Y; that is,

$$\forall U \subseteq X (U \in \tau_X \implies f[U] \in \tau_Y).$$

Example

Consider the function

$$f : \mathbb{R} \to \mathbb{R}$$

given by setting

$$f(x) = \begin{cases} x & \textit{if } x \leqslant 0 \\ 0 & \textit{otherwise} \end{cases}$$

We show that f is continuous but not open (see blackboard).

That’s it for today, any questions?