Topology project, 3rd lecture

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Plan for the day

- Recap
- New stuff
- Break from 11h45-12h
- More new stuff

Recap: generating new topologies

Definition (subspace)

Let (X, τ) be a ts and $S \subseteq X$. We denote by τ_S the *subspace topology on* S defined as

$$\tau_S := \{U \cap S \mid U \in \tau\}.$$

Definition (finite product top)

Let X and Y be ts. We define a topology on the product $X \times Y$, called the *product topology*, as follows: a set $U_0 \times U_1 \subseteq X \times Y$ is basic open :iff U_0 is open in X and U_1 is open in Y.

Lemmas: Both constructions can be obtained by taking bases for original space(s).

Proposition: The constructions *commute*.

Recap: closed sets, closure and interior

Definition

Let (X, τ) be a ts. We say that a set $U \in \mathcal{P}(X)$ is *closed* if its complement is open; i.e., if $(X - U) \in \tau$.

Definition

Let X be a ts and $S \subseteq X$. We denote by

$$cl(S) := \overline{S} := \bigcap\{S \subseteq C \mid C \text{ is closed}\}$$

the *closure of* S, which is the smallest closed set K such that $S \subseteq K$.
We denote by

$$int(S) := \bigcup\{U \subseteq S \mid U \text{ is open}\}$$

the *interior of* S, which is the largest open set K such that $K \subseteq S$.

Observation: Using this def., we get:

(C) $S \subseteq X$ is closed iff $S = \overline{S}$;
(O) $S \subseteq X$ is open iff $S = int(S)$.

Recap: neighbourhoods

Definition

Given a ts (X, τ) and $x \in X$, $V \subseteq X$ is a *neighbourhood* of x :iff there is an open set U such that $x \in U \subseteq V$.
Observe that if a neighbourhood V of a point x is open, the definition simplifies: V is an open neighbourhood of a point x iff $x \in V$ and V is open.

Proposition

Suppose X is a ts and $S \subseteq X$. Then TFAE for a point $x \in X$:

- x is in the closure of S; i.e., $x \in cl(S)$.
- All open neighbourhoods U of x have non-empty intersection with S; i.e., $U \cap S \neq \varnothing$.

Recap: summary of epistemic intuition

Logic	Topology
Epistemic worlds/situations/etc.	Points, $x \in X$
Verifiable propositions	Open sets, $U \in \tau$
Falsifiable propositions	Closed sets, $U^C \in \tau$
Verifiable propositions true at x	Open neighbourhoods U of x
(Sub)basic verifiable propositions	(Sub)basic opens

Continuity

Recap: continuous and open maps (and why the latter do not formalise continuity)

Definition (Continuous map)

Let $(X, \tau_X), (Y, \tau_Y)$ be ts and $f : X \to Y$ a map between them. Then f is *continuous* :iff for all $U \subseteq Y$ open in Y, the preimage $f^{-1}[U] := \{x \in X \mid f(x) \in U\}$ is open in X; i.e.,

$$\forall U \subseteq Y (U \in \tau_Y \implies f^{-1}[U] \in \tau_X).$$

Definition (Open map)

Let (X, τ_X) and (Y, τ_Y) be ts, and $f : X \to Y$ a map between them. We say that f is *open* if for every open U in X, its image $f[U] = \{f(x) \in Y \mid x \in U\}$ is open in Y; that is,

$$\forall U \subseteq X (U \in \tau_X \implies f[U] \in \tau_Y).$$

Example

Consider the function

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) \mapsto \begin{cases} x & \textit{if } x \leqslant 0 \\ 0 & \textit{otherwise} \end{cases}$$

We showed that f is continuous but not open.

Continuous functions

Definition

Let $(X, \tau_X), (Y, \tau_Y)$ be ts and $f : X \to Y$ a map between them. Then f is *continuous* :iff for all $U \subseteq Y$ open in Y, the preimage $f^{-1}[U] := \{x \in X \mid f(x) \in U\}$ is open in X; i.e.,

$$\forall U \subseteq Y (U \in \tau_Y \implies f^{-1}[U] \in \tau_X).$$

Proposition

Let $f : X \to Y$ be a map between topological spaces. Then TFAE:

(i) f is continuous

(ii) For every $S \subseteq X$: $f\left(\overline{S}\right) \subseteq \overline{f(S)}$, i.e., if $x \in cl_X(S)$ then $f(x) \in cl_Y(f(S))$

Interpretation: For $S \subseteq X$ and $x \in X$, we say that x *is close to S* :iff $x \in cl(S)$.
Then f is continuous iff

for every $S \subseteq X$, f maps points close to S to points close to $f(S)$.

Proof.

See blackboard. □

More equivalent definitions of contintuity

Proposition

Let $f : X \to Y$ be a map between topological spaces and $\mathcal{B}_Y$ a (sub)basis for the topology on Y. Then the following are equivalent:

1. f is continuous.
2. For every (sub)basic open $U \in \mathcal{B}_Y$, its preimage $f^{-1}[U]$ is open in X.
3. For every closed set U in Y, its preimage $f^{-1}[U]$ is closed in X.
4. For every $x \in X$, whenever $V \subseteq Y$ is a (basic) open neighbourhood of $f(x)$, there is an open neighbourhood $U \subseteq X$ of x such that $f[U] \subseteq V$.

Proof.

We covered $1. \Leftrightarrow 2.$ on Friday, the remaining is left as an exercise. □

Remark: f is said to be *continuous at a point* $x \in X$ if condition 4. holds for x.

You should show that under the "close to"-interpretation, we have that f is continuous at a point $x \in X$ iff

$(*)_{local}$ for every $S \subseteq X$, if x is close to S then $f(x)$ is close to $f[S]$.

Continuous maps between S4 frames

- Topological spaces are much more and much else than $\mathbb{R}$; likewise must the top. notion of continuity cover much more and much else than continuity on $\mathbb{R}$.
- What are the continuous maps on reflexive and transitive Kripke frames?

Definition

Let $\mathfrak{F} = (W, R), \mathfrak{F}' = (W', R')$ be Kripke frames. A map $f : W \to W'$ satisfies

- *the forth condition* if whenever xRy, we have $f(x)R'f(y)$; and
- *the back condition* if whenever $f(x)R'y'$, $\exists y \in W$ s.t. xRy and $f(y) = y'$.

Proposition

Let $\mathfrak{F} = (W, R)$ and $\mathfrak{F}' = (W', R')$ be two reflexive and transitive Kripke frames, equipped with the Alexandroff topology, and $f : W \to W'$ a map between them. Then:

1. f satisfies the forth condition if and only if f is continuous.
2. f satisfies the back condition if and only if f is open.

Proof.

See blackboard. □

Example of open, but not continuous map

See blackboard.

Homeomorphisms, embeddings, and quotient maps

Definition

Let $f : X \to Y$ be a map between ts. We say that f is

- a *quotient map* if (i) it is surjective and (ii) for all $U \subseteq Y$,
  $$U \text{ is open in } Y \text{ iff } f^{-1}(U) \text{ is open in } X;$$
- a *homeomorphism* if it is bijective, continuous and open; and
- a *(topological) embedding* or an *interior map* if the restriction
  $$f' : X \to f[X]$$
  is a homeomorphism (where $f[X] \subseteq Y$ has the subspace topology).

Important: Homeomorphism is the topological version of an "isomorphism": Whenever topological spaces are homeomorphic, they are topologically the same (i.e., have the same top. properties).

Characterizing embeddings and quotient maps

Definition

Let $f : X \to Y$ be a map between ts. Then f is *closed* if for every closed U in X, its image $f[U] = \{f(x) \in Y \mid x \in U\}$ is closed in Y.

Lemma

Let $f : X \to Y$ be a map between ts. Then:

(0.q) f is a quotient map if and only if (i) f is surjective and (ii') for all $U \subseteq Y$, U is closed in Y if and only if $f^{-1}(U)$ is closed in X.

(1.q) If f is a quotient map, then f is surjective and continuous.

(2.q) If f is surjective, continuous and open, then f is a quotient map.

(3.q) If f is surjective, continuous and closed, then f is a quotient map.

(1.e) If f is an embedding, then f is injective and continuous.

(2.e) If f is injective, continuous and open, then f is an embedding.

(3.e) If f is injective, continuous and closed, then f is an embedding.

Proof.

(0.q)-(3.q) follow almost directly by definition. (3.e) matches a HW exercise. So we show (1.e) and (2.e) (see blackboard). □

"Quotienting is like gluing"

Definition (quotient topology)

Let X be a ts, $\sim$ an equivalence class on X, and

$$q : X \to X/\sim, x \mapsto [x]_\sim$$

The *quotient topology* on $X/\sim$ is defined as follows:

$U \subseteq X/\sim$ is open :iff $q^{-1}[U]$ is open in X.

Gluing endpoints of an interval to obtain a circle

See blackboard.

Some preliminaries for Tuesday and Wednesday

Filters and filter bases

Definition (Filter and filter base)

Let X be a set. A collection of subsets $F \subseteq (\mathcal{P}(X) - \{\varnothing\})$ is a *filter base* :iff

- $X \in F$;
- If $A, B \in F$ then $A \cap B \in F$.

We say that a filter base is a *filter* if it is upwards closed:

- If $A \in F$ and $A \subseteq B$, then $B \in F$.

Useful fact

Let X be a set, and F a filter base. Then the *upwards closure of F*

$$F^{\uparrow} := \{C \subseteq X : \exists G \in F, G \subseteq C\},$$

is a filter.

Given a ts (X, τ) and $x \in X$, we denote the set of neighbourhoods of x by $\mathcal{N}(x)$.

Lemma

Let (X, τ) be a ts and $x \in X$. Then $\mathcal{N}(x)$ is a filter.

Proof.

See blackboard. □

Convergence and filter (bases)

Definition

Let (X, τ) be a topological space and $F \subseteq \tau$ a filter (base). We say that the filter (base) F *converges to a point* x, and that x is a *limit of the filter (base)*, if and only if for every $U \in \mathcal{N}(x)$, there is some $V \in F$ such that $V \subseteq U$.

Note that the notion of convergence *does not say anything about uniqueness*.

That’s it for today. Please read section 4.1 and 4.2 to prepare for tomorrow’s tutorial. Any questions?